A Type Theory for Parameterised Spectra

Mitchell Riley

12$^{\text{th}}$ February 2020

Some toposes:

- Set
- Sheaves on a space – algebraic geometry
- The effective topos – computability

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Type Theory

M is a finitely generated module

Type Theory for the Working Mathematician

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M is a finitely generated module

Set
M is an ordinary finitely generated module

Type Theory for the Working Mathematician

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$$\begin{tikzcd}[column sep=tiny]
& \begin{array}{c}\text{Type Theory}\\ \textit{M is a finitely}\\ \textit{generated module}\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\text{Set}\\ \textit{M is an ordinary}\\ \textit{finitely generated module}\end{array} & & \begin{array}{c}\text{Sheaves on } X\\ \textit{M is a sheaf of modules}\\ \textit{of finite type}\end{array}
\end{tikzcd}$$

Type Theory for the Working Mathematician

Some ∞-toposes:

- Spaces – homotopy theory
- ∞-sheaves – derived algebraic geometry
- Smooth spaces – synthetic differential geometry

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- Spaces – homotopy theory
- ∞-sheaves – derived algebraic geometry
- Smooth spaces – synthetic differential geometry
- Parameterised spectra – stable homotopy theory

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Overview

- Doing mathematics in type theory
- Spectra and the ∞-topos of parameterised spectra
- A type theory that interprets into parameterised spectra

Type Theories

Products

Suppose we have two arbitrary sets A and B. We can build a function, say

$$f : A \times B \to A \times (B \times A)$$

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Type theory lets us mechanically convert from the former version to the latter.

Judgements and Rules

- If 'A type', then A is an object of the category
- If '$\Gamma \vdash a : A$', where Γ is a list of assumptions

  $$x_1 : X_1, x_2 : X_2, \ldots, x_n : X_n$$

  then there is a map

  $$a : X_1 \times X_2 \times \cdots \times X_n \to A$$

The rules look like:

$$\textsc{rule-name}\ \frac{\mathcal{J}_1 \qquad \ldots \qquad \mathcal{J}_n \qquad \text{(premises)}}{\mathcal{J} \qquad \text{(conclusion)}}$$

Products

$$\text{VAR}\ \frac{}{\Gamma, x : A, \Gamma' \vdash x : A}$$

Products

$$\textsc{var}\ \frac{}{\Gamma, x : A, \Gamma' \vdash x : A}$$

$$\times\textsc{-form}\ \frac{A \text{ type} \qquad B \text{ type}}{A \times B \text{ type}}$$

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$$\times\text{-}\textsc{elim}\ \frac{\Gamma \vdash p : A \times B \qquad \Gamma, x : A, y : B \vdash c : C}{\Gamma \vdash \mathsf{let}\,(x, y) = p \,\mathsf{in}\, c : C}$$

and some equations.

Products

The function from before:

$$
\dfrac{\dfrac{x : A, y : B \vdash x : A \qquad \dfrac{x : A, y : B \vdash y : B \qquad x : A, y : B \vdash x : A}{x : A, y : B \vdash (y, x) : B \times A}\ \times\text{-INTRO}}{p : A \times B, x : A, y : B \vdash (x, (y, x)) : A \times (B \times A)}\ \times\text{-INTRO}}{p : A \times B \vdash \mathsf{let}\,(x, y) = p\,\mathsf{in}\,(x, (y, x)) : A \times (B \times A)}\ \times\text{-ELIM}
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Theorem

The rules on the previous slide present the free category-with-products on a set of objects.

Functions

For any two sets A, B, there is a set of functions $A \to B$.

$$\to\text{-FORM}\ \frac{A \text{ type} \qquad B \text{ type}}{A \to B \text{ type}}$$

$$\to\text{-INTRO}\ \frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.b : A \to B} \qquad \to\text{-ELIM}\ \frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B}$$

(and again some equations)

This kind of *exponential object* exists in any *cartesian closed category* (sets, nice spaces, sheaves, . . .).

Dependent Type Theory

Type theory can be made more powerful by allowing types to *depend* on terms.

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Example

Each point of a differentiable manifold has a tangent space:

$$x : M \vdash T_x M \text{ type}$$

Dependent Type Theory

The product type can be generalised to *dependent* pairs:

$$\Sigma\text{-}\textsc{form}\ \frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash (x : A) \times B(x) \text{ type}}$$

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The dependent pair type $(x : M) \times T_x M$ is the tangent bundle TM.

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Similarly for *dependent* functions:

$$\Pi\text{-}\textsc{form}\ \frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash (x : A) \to B(x) \text{ type}}$$

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The dependent function type $(x : M) \to T_x M$ is a vector field.
(sort of, one would need to think carefully about continuity)

Homotopy Type Theory

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$$\text{Path-}\textsc{elim}\ \frac{\begin{array}{c}\Gamma, x : A, x' : A, z : \mathsf{Path}_A(x, x') \vdash C\ \mathsf{type} \\ \Gamma, x : A \vdash c : C[x/x', \mathsf{refl}_x/p] \\ \Gamma \vdash p : \mathsf{Path}_A(a, a')\end{array}}{\Gamma \vdash \mathsf{ind}(z.c, a, a', p) : C[a/x, a'/x', p/z]}$$

Homotopy Type Theory

Definition

A type A is *contractible* if there is a term of the type

$$\mathrm{isContr}(A) := (c : A) \times ((x : A) \to \mathsf{Path}_A(c, x))$$

(Don't worry, this doesn't mean just path-connected!)

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The *homotopy fiber* of a function $f : A \to B$ over a point $b : B$ is

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Definition

A function is an *equivalence* if the homotopy fiber over every point is contractible:

$$\mathrm{isEquiv}(f) := (b : B) \to \mathrm{isContr}(\mathrm{hfib}_f(b))$$

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Some synthetic results:

- Some homotopy groups of spheres (Shulman, Brunerie, Licata)
- Freudenthal Suspension Theorem (Lumsdaine, Licata)
- Localisation (Chrsitensen, Opie, Rijke, Scoccola)
- Blakers–Massey Theorem (Anel, Biedermann, Finster, Joyal)
- Serre Spectral Sequence (Avigad, Awodey, Buchholtz, Rijke, Shulman, van Doorn)

Homotopy Type Theory

Theorem (Kapulkin and Lumsdaine 2012, after Voevodsky)

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Types in the theory become categorical constructions:

Γ ctx	Objects Γ
$\Gamma \vdash A$ type	Fibrations $A \twoheadrightarrow \Gamma$
Σ and Π types	Adjoints to pullback functors $\mathcal{C}/\Gamma \to \mathcal{C}/A$
Path types	Path space fibration
...	...

Spectra and Parameterised Spectra

Motivating Spectra

Theorem

Singular cohomology is representable: for any abelian group G and pointed CW-complex X,

$$\tilde{H}^n(X; G) \cong [X, K(G, n)]_{\mathrm{pt}}$$

where $K(G, n)$ is an Eilenberg-MacLane space.

Motivating Spectra

Definition (Eilenberg–Steenrod axioms)

A *reduced cohomology theory* is a sequence of functors

$$\tilde{E}^n : \begin{pmatrix} \text{pointed connected CW-complexes} \\ \text{up to homotopy} \end{pmatrix}^{\mathrm{op}} \to (\text{abelian groups})$$

such that

1. Wedge sums are taken to products; and,
2. For each CW-pair (X, A), the sequence
$$\tilde{E}^n(X/A) \to \tilde{E}^n(X) \to \tilde{E}^n(A)$$
is exact.
3. There is a natural isomorphism $\tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X)$.

Motivating Spectra

Theorem (Brown Representability)

For any reduced cohomology theory $\tilde{E}^*$, there is a sequence of pointed connected CW-complexes K_n so that

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A sequence $\{K_n\}$ does not quite determine a cohomology theory by itself: we are missing the suspension isomorphisms.

Motivating Spectra

For any X we have a natural isomorphism:

$$[X, K_n]_{\mathrm{pt}} \cong \tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X) \cong [\Sigma X, K_{n+1}]_{\mathrm{pt}} \cong [X, \Omega K_{n+1}]_{\mathrm{pt}}$$

The image of the identity map on K_n is a map $\alpha_n : K_n \to \Omega K_{n+1}$.

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Letting X vary over the spheres S^k, we see in fact α_n is a weak equivalence.

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Definition

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Example

The *zero spectrum* with $K_n = \{\star\}$.

Example

The *sphere spectrum*, whose homotopy groups are the stable homotopy groups of spheres

Option 1: Spectra in Type Theory

Spectra can be defined almost verbatim in HoTT:

$$\begin{aligned}\mathrm{Spectrum} :&= (K : (\mathbb{N} \to \mathrm{PtdType})) \\ &\times ((n : \mathbb{N}) \to \mathrm{Equiv}(K(n), \Omega K(n+1)))\end{aligned}$$

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An important operation on spectra is the *smash product*. Recall the smash product of pointed spaces:

$$A \wedge B := (A \times B)/(A \vee B)$$

Even showing this is associative in HoTT is a task! (van Doorn 2018)

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Idea: Instead of rebuilding spectra inside type theory, model type theory in a category where they already exist.

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Idea: Instead of rebuilding spectra inside type theory, model type theory in a category where they already exist.

The category of spectra is lousy for modelling type theory.

- Yes: dependent pair type, path type
- No: everything else

Option 2: A Model in Parameterised Spectra

“Definition”

A *parameterised spectrum* is a bundle of spectra over a space.

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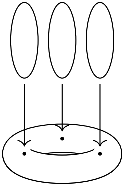

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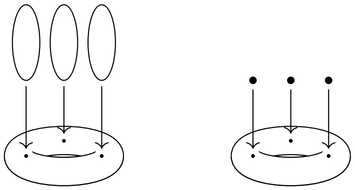

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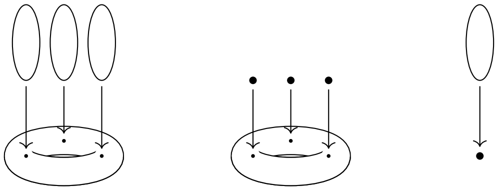

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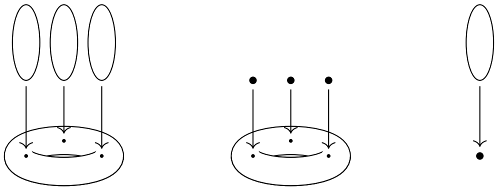

Theorem (Joyal 2008)

The ∞-category of parameterised spectra, $P\mathrm{Spec}$, is an ∞-topos.

So is a model of HoTT.

A Type Theory for Parameterised Spectra

Stable Homotopy Type Theory?

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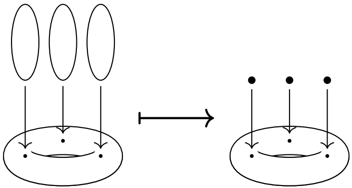
We need to figure out how to add new type formers that give access to that structure.

Underlying Space

For every type A there should be a type $\natural A$ that deletes the spectral information.



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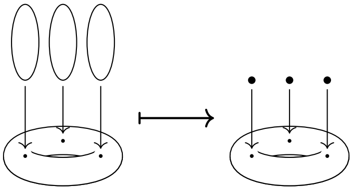

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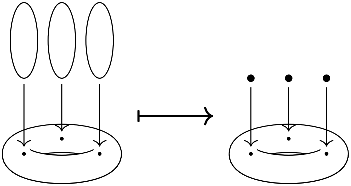

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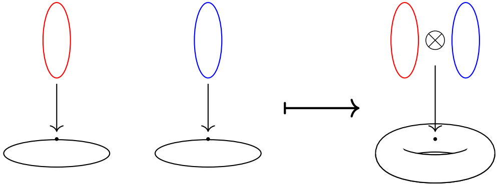
Smash Product

For two types A and B, there should be a type $A \otimes B$ that corresponding to the 'external smash product'.



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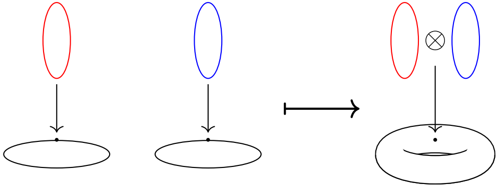

$$\otimes\text{-INTRO}\ \frac{\Gamma^0, \Omega, \Omega'^0, \Gamma'^0 \vdash a : A \qquad \Gamma^0, \Omega^0, \Omega', \Gamma'^0 \vdash b : B}{\Gamma, (\Omega)(\Omega'), \Gamma' \vdash a \otimes b : A \otimes B}$$

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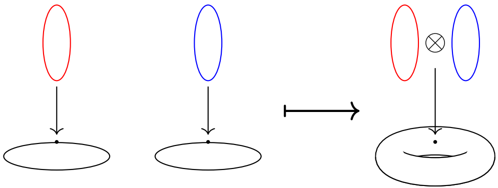

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$$\otimes\text{-ELIM}\ \frac{\begin{array}{c}\Gamma, z : A \otimes B \vdash C \text{ type} \\ \Gamma, (x : A)(y : B) \vdash c : C[x \otimes y/z] \\ \Gamma \vdash s : A \otimes B\end{array}}{\Gamma \vdash \text{let } x \otimes y = s \text{ in } c : C[s/z]}$$

More Formers

- The sphere spectrum $\mathbb{S}$: the monoidal unit for $\otimes$

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- Hom types $A \multimap B$: right adjoint to $- \otimes A$

Progress

What's done:

- Judgemental structure
- Type formers and their interactions with the context

Combining dependent types and 'linear' features is difficult! And interesting!

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What's left:

- Check all the admissible rules work
- Actually use it!
- Describe intended semantics more precisely
- Code up a type-checker?

References I

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