A Type Theory for Parameterised Spectra

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8th March 2020

Spectra and Parameterised Spectra

Spectra

Definition

A *prespectrum* E is a sequence of pointed types $E : \mathbb{N} \to \mathcal{U}_\star$ together with pointed maps $\alpha_n : E_n \to_\star \Omega E_{n+1}$.

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The *sphere prespectrum* has $E_n :\equiv S^n$, with α_n the transpose of $\Sigma S^n \to_\star S^{n+1}$

Cohomology and Homology

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Given a spectrum E and a pointed type X,

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$$E_n(X) :\equiv \operatorname*{colim}_{k \to \infty} \pi_{n+k}(X \wedge E_k)$$

where $A \wedge B := (A \times B)/(A \vee B)$ is the smash product.

Problem

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Instead: Model type theory in a topos where spectra already exist.

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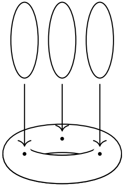

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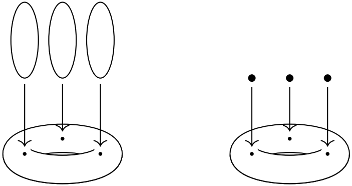

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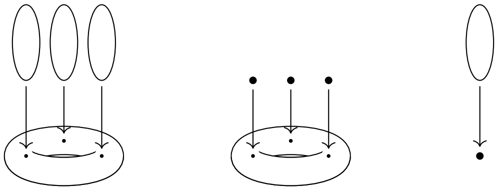

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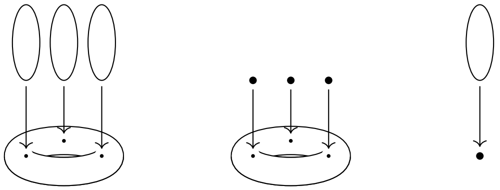

Theorem (Joyal 2008)

The ∞-category of parameterised spectra, $P\mathrm{Spec}$, is an ∞-topos.

So is a model of HoTT.

A Toy Model: Families of Pointed Types

Definition

A *context* Γ is a type Γ_B and a type family $\Gamma_E : \Gamma_B \to \mathcal{U}$ with a chosen basepoint $\gamma_0(\gamma) : \Gamma_E(\gamma)$ for each $\gamma : \Gamma_B$

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$$\begin{array}{ccccc} \Gamma_E, & \leftarrow & A_E, & \leftarrow & B_E, \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_B, & \leftarrow & A_B, & \leftarrow & B_B, \end{array}$$

(This was one of Ulrik's 'toy models' of cohesion)

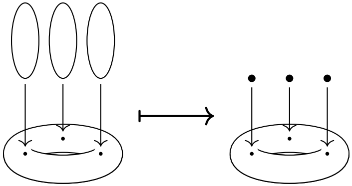
Goal:

Add type formers that capture some of the additional structure in these models.

The ‘Underlying Space’ Modality

Underlying Space

For every type A there should be a type $\natural A$ that deletes the spectral information.



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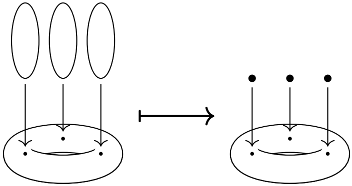

This $\natural$ is an idempotent monad and comonad that is adjoint to itself.

Like Mike's Spatial Type Theory, but with $\sharp \equiv \flat$.

Recall: Spatial Type Theory

$\flat$ is a lex idempotent comonad, $\sharp$ is an idempotent monad, and $\flat \dashv \sharp$.

We put in a judgemental version of $\flat$ and have the type formers interact with it.

$$\Delta \mid \Gamma \vdash a : A \qquad \text{corresponds to} \qquad a : \flat\Delta \times \Gamma \to A$$

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$$\flat\textsc{-intro}\ \frac{\Delta \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^\flat : \flat A}$$

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$$\flat\Delta \times \Gamma \to \flat\Delta \to \flat\flat\Delta \to \flat A$$

The Unit?

In spatial type theory, the counit is invisible: there was an admissible rule

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With $\natural$ we have a dilemma: there is both a unit $A \to \natural A$ and a counit $\natural A \to A$, the round trip on A is not the identity.

$$\textsc{unit}?\ \frac{\Delta, x : A \mid \Gamma \vdash b : B}{\Delta \mid x : A, \Gamma \vdash b : B}$$

We choose to make the *counit* explicit.

Zones?

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$$x : A, y : B(x) \mid z : C \vdash d : D$$

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What if we want to precompose with the unit on $x : A$ only?

$$y : B(x) \mid x : A, z : C \vdash d : D$$

Zeroed Variables

$$\frac{\Gamma\ \mathsf{ctx} \qquad \Gamma^0 \vdash A\ \mathsf{type}}{\Gamma, x^0 : A\ \mathsf{ctx}}$$

$$\frac{}{\Gamma, x^0 : A, \Gamma' \vdash x^0 : A} \qquad \frac{}{\Gamma, x : A, \Gamma' \vdash x^0 : A^0}$$

Γ^0 denotes an operation that zeroes all the variables in Γ.

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$$\textsc{counit}\ \frac{\Gamma, x : A, \Gamma' \vdash b : B}{\Gamma, x^0 : A, \Gamma'[x^0/x] \vdash b[x^0/x] : B[x^0/x]}$$

$$\textsc{unit}\ \frac{\Gamma, x^0 : A, \Gamma' \vdash b : B}{\Gamma, x : A, \Gamma' \vdash b : B}$$

Rules for ♮

$$\natural\text{-}\textsc{form}\ \frac{\Gamma^0 \vdash A \text{ type}}{\Gamma \vdash \natural A \text{ type}}$$

$$\natural\text{-}\textsc{intro}\ \frac{\Gamma^0 \vdash a : A}{\Gamma \vdash a^\natural : \natural A} \qquad \natural\text{-}\textsc{elim}\ \frac{\Gamma \vdash a : \natural A}{\Gamma \vdash a_\natural : A}$$

$$a^\natural{}_\natural \equiv a \qquad n \equiv n^0{}_\natural{}^\natural$$

These are the ♯-style rules. The ♭-style rules are derivable!

$\natural$ and Dependency

A context

$$x : A, y^0 : B(x^0), z : C(x, y^0), w^0 : D(x^0, y^0, z^0)$$

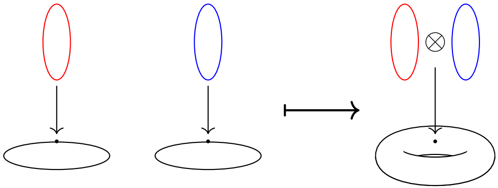
corresponds in the model to

$$\begin{array}{ccccccc}
x : A_E, & \dashleftarrow & & & z : C_E(x) & & \\
\downarrow & & & & \downarrow & & \\
x^0 : A_B, & \dashleftarrow & y^0 : B_B(x^0), & \dashleftarrow & z^0 : C_B(x^0, y^0), & \dashleftarrow & w^0 : D_B(x^0, y^0, z^0)
\end{array}$$

The Smash Product

Smash Product

For two types A and B there should be a type $A \otimes B$ corresponding to the 'external smash product'.



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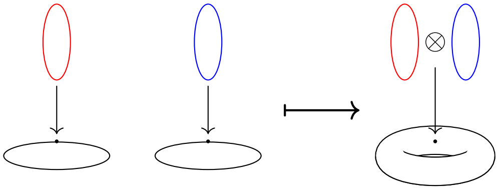

This is a symmetric monoidal product with no additional structural rules.

Bunched Contexts

We can take a cue from ‘bunched logics’, where there are two ways of combining contexts, an ordinary cartesian one and a linear one.

$$\frac{\Gamma_1 \text{ ctx} \qquad \Gamma_2 \text{ ctx}}{\Gamma_1, \Gamma_2 \text{ ctx}} \qquad\qquad \frac{\Gamma_1 \text{ ctx} \qquad \Gamma_2 \text{ ctx}}{(\Gamma_1)(\Gamma_2) \text{ ctx}}$$

For the comma *only*, we have weakening and contraction as normal.

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A typical context:

$$x : A, (y : B)(z : C, (p : P)(q : Q)), w : D$$

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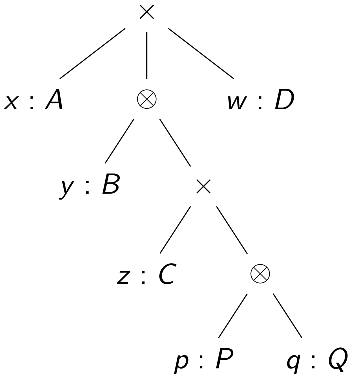
Or as a tree:



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$$\otimes\text{-INTRO}\ \frac{\Omega \vdash a : A \qquad \Omega' \vdash b : B}{(\Omega)(\Omega') \vdash a \otimes b : A \otimes B}$$

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$$\otimes\text{-ELIM}\;\frac{\begin{array}{c}\Gamma\{(x : A)(y : B)\} \vdash c : C \\ \Delta \vdash s : A \otimes B\end{array}}{\Gamma\{\Delta\} \vdash \text{let } x \otimes y = s \text{ in } c : C}$$

Smash and Dependency

When does a 'dependent external smash' $A \otimes B$ make sense?

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We can only do it when the fibers of A and B don't depend on Γ_E.

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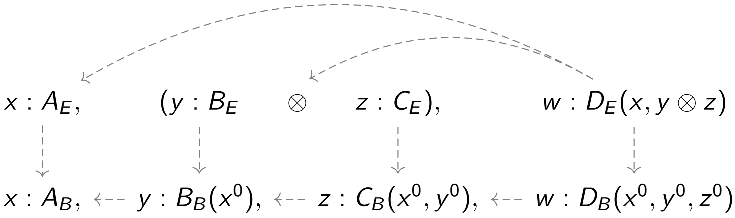

Dependent Smash

The judgemental ‘context smash’.

$$\frac{\Gamma \vdash \Omega \ \mathsf{tele}}{\Gamma, \Omega \ \mathsf{ctx}} \qquad \frac{\Gamma \vdash \Omega \ \mathsf{tele} \quad \Gamma, \Omega \vdash \Omega' \ \mathsf{tele}}{\Gamma \vdash \Omega, \Omega' \ \mathsf{tele}} \qquad \frac{\Gamma^0 \vdash \Omega \ \mathsf{tele} \quad \Gamma^0, \Omega^0 \vdash \Omega' \ \mathsf{tele}}{\Gamma \vdash (\Omega)(\Omega') \ \mathsf{tele}}$$

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A type that internalises it:

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$$\otimes\text{-INTRO} \; \frac{\Gamma^0, \Omega, \Omega'^0, \Gamma'^0 \vdash a : A \qquad \Gamma^0, \Omega^0, \Omega', \Gamma'^0 \vdash b : B[a^0/x^0]}{\Gamma, (\Omega)(\Omega'), \Gamma' \vdash a \otimes b : \bigcirc\!\!\!\!\!\sum_{(x^0 : A)} B}$$

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- Dependent 'linear hom' types $A \multimap B$, right adjoint to $- \otimes A$.

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$$(x : A)(y : B(x^0)), z : C(x, y), w^0 : D(x^0, y^0, z^0)$$

as

$$\color{red}{x} : A, \color{blue}{y} : B(x), \color{purple}{z} : C(\color{red}{x}, \color{blue}{y}), w : D(x, y, z)$$

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In $\otimes$-elim, we create two new colours that sum to the colour of the target:

$$\text{let } x \otimes y = p \text{ in } c$$

Eg: Uniqueness principle for $\otimes$

Proposition

Suppose A and B are types. If $C : A \otimes B \to \mathcal{U}$ is a type family and $f : \prod_{(p:A\otimes B)} C(p)$, then for any $p : A \otimes B$ we have

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Let $P : A \otimes B \to \mathcal{U}$ denote the type family

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We wish to find an element of $\prod_{(p:A\otimes B)} P(p)$. By $\otimes$-induction we may assume $p \equiv x' \otimes y'$. Our goal is now

$$(\text{let } x \otimes y = x' \otimes y' \text{ in } f(x \otimes y)) = f(x' \otimes y')$$

Which by the β-rule reduces to $f(x' \otimes y') = f(x' \otimes y')$. $\square$

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But we *can* form $a \otimes a : A \otimes A$, the diagonal map on the base and constantly zero in the fibers.

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Proof.

Given $w : \natural \bigcirc\!\!\!\!\!\sum_{(x:A)} B(x)$ we have a term $w_\natural : \bigcirc\!\!\!\!\!\sum_{(x:A)} B(x)$. Induction on this gives $x : A$ and $y : B(x)$, from which we can produce $(x^\natural, y^\natural) : \sum_{(u:\natural A)} \natural B(u_\natural)$.

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$\natural \mathop{\bigcirc\kern-0.9em\sum}_{(x:A)} B(x) \simeq \sum_{(u:\natural A)} \natural B(u_\natural)$

Proof.

Given $w : \natural \mathop{\bigcirc\kern-0.9em\sum}_{(x:A)} B(x)$ we have a term $w_\natural : \mathop{\bigcirc\kern-0.9em\sum}_{(x:A)} B(x)$. Induction on this gives $x : A$ and $y : B(x)$, from which we can produce $(x^\natural, y^\natural) : \sum_{(u:\natural A)} \natural B(u_\natural)$.

In the other direction, from $z : \sum_{(x:\natural A)} \natural B(x_\natural)$ we get $\mathrm{pr}_1(z)_\natural : A$ and $\mathrm{pr}_2(z)_\natural : B(\mathrm{pr}_1(z)_\natural)$. These terms are (vacuously) blue and red respectively so we can form

$$(\mathrm{pr}_1(z)_\natural \otimes \mathrm{pr}_2(z)_\natural) : \mathop{\bigcirc\kern-0.9em\sum}_{x:A} B(x)$$

and apply $(-)^\natural$. Now check round trips. □

Axioms?

How do we characterise parameterised spectra amongst the models? Possibilities:

- $\natural$ is $\mathbb{S}$-nullification
- $\Sigma^n\mathbb{S} \to \Omega\Sigma^{n+1}\mathbb{S}$ is an equivalence
- Relate $\mathbb{S}$ to the stable homotopy groups of ordinary spheres

References I

Joyal, André (2008). *Notes on logoi*. URL: http://www.math.uchicago.edu/~may/IMA/JOYAL/Joyal.pdf.

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