Commuting Cohesions

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Spatial Type Theory

Spatial type theory is an extension of HoTT whose intended models are 'local toposes':

$$\begin{array}{c} \mathcal{E} \\ \text{Disc} \Big\uparrow \dashv \Big\downarrow \Gamma \dashv \Big\uparrow \text{CoDisc} \\ \mathcal{S} \end{array}$$

with the outer functors fully faithful.

- $\flat :\equiv \mathrm{Disc} \circ \Gamma$ is a lex idempotent comonad,
- $\sharp :\equiv \mathrm{CoDisc} \circ \Gamma$ is an idempotent monad,
- with $\flat \dashv \sharp$.

In nice settings, there is a type G that "detects connectivity"

$$\{X \text{ is } \flat\text{-modal}\} \longleftrightarrow \{X \text{ is } G\text{-null}\}$$

Then $\int :\equiv$ (nullification at G) is left adjoint to $\flat$.

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Examples of Cohesion

"Topological" ∞-groupoids (say, sheaves on Cartesian spaces):

- $\int X$: Fundamental ∞-groupoid, topologised discretely
- $\flat X$: Discrete retopologization
- $\sharp X$: Codiscrete retopologization
- Connectivity detected by $\mathbb{R}$

Simplicial ∞-groupoids:

- $\mathsf{re}\, X$: Realization, as a 0-skeletal simplicial ∞-groupoid
- $\mathsf{sk}_0\, X$: 0-skeleton
- $\mathsf{csk}_0\, X$: 0-coskeleton
- Connectivity detected by $\Delta[1]$ (postulated as a total order with 0 and 1)

From $\Delta[1]$ you can define $\Delta[n] :\equiv$ (chains of length n in $\Delta[1]$) and $X_n :\equiv \mathsf{sk}_0(\Delta[n] \to X)\ldots$

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Spatial Type Theory

$$\textsc{ctx-empty}\ \frac{}{\cdot \mid \cdot\ \mathsf{ctx}}$$

$$\textsc{ctx-ext-crisp}\ \frac{\Delta \mid \cdot \vdash A\ \mathsf{type}}{\Delta, x : A \mid \cdot\ \mathsf{ctx}} \qquad \textsc{ctx-ext}\ \frac{\Delta \mid \Gamma \vdash A\ \mathsf{type}}{\Delta \mid \Gamma, x : A\ \mathsf{ctx}}$$

$$\textsc{var-crisp}\ \frac{}{\Delta, x : A, \Delta' \mid \Gamma \vdash x : A}$$

$$\textsc{var}\ \frac{}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A}$$

Spatial Type Theory

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$$\textsc{ctx-ext-}\heartsuit\ \frac{\heartsuit \setminus \Gamma \vdash A\ \mathsf{type}}{\Gamma, x :_{\heartsuit} A\ \mathsf{ctx}} \qquad\qquad \textsc{var-}\heartsuit\ \frac{}{\Gamma, x :_{\heartsuit} A, \Gamma' \vdash x : A}$$

Definition. $\heartsuit \setminus \Gamma$ deletes all variables *not* annotated by $\heartsuit$.

In the dual context formulation, $\Delta \mid \Gamma\ \mathsf{ctx} \rightsquigarrow \Delta \mid \cdot\ \mathsf{ctx}$.

Spatial Type Theory

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$\flat$

$$\flat\text{-}\textsc{Form}\ \frac{\heartsuit \setminus \Gamma \vdash A \ \mathsf{type}}{\Gamma \vdash \flat_{\heartsuit} A \ \mathsf{type}}$$

$$\flat\text{-}\textsc{Intro}\ \frac{\heartsuit \setminus \Gamma \vdash M : A}{\Gamma \vdash M^{\flat_{\heartsuit}} : \flat_{\heartsuit} A}$$

$$\flat\text{-}\textsc{Elim}\ \frac{\begin{array}{cc} \heartsuit \setminus \Gamma \vdash A \ \mathsf{type} & \Gamma, x : \flat_{\heartsuit} A \vdash C \ \mathsf{type} \\ \Gamma \vdash M : \flat_{\heartsuit} A & \Gamma, u :_{\heartsuit} A \vdash N : C[u^{\flat_{\heartsuit}}/x] \end{array}}{\Gamma \vdash (\mathsf{let}\ u^{\flat_{\heartsuit}} := M \ \mathsf{in}\ N) : C[M/x]}$$

$$\sharp\text{-FORM}\ \frac{\heartsuit\Gamma \vdash A\ \mathsf{type}}{\Gamma \vdash \sharp_{\heartsuit} A\ \mathsf{type}}$$

$$\sharp\text{-INTRO}\ \frac{\heartsuit\Gamma \vdash M : A}{\Gamma \vdash M^{\sharp\heartsuit} : \sharp_{\heartsuit} A} \qquad \sharp\text{-ELIM}\ \frac{\heartsuit \setminus \Gamma \vdash N : \sharp_{\heartsuit} A}{\Gamma \vdash N_{\sharp_{\heartsuit}} : A}$$

Definition. $\heartsuit\Gamma$ adds the $\heartsuit$ annotation to every variable in Γ.

With dual contexts, $\Delta \mid \Gamma\ \mathsf{ctx} \rightsquigarrow \Delta, \Gamma \mid \cdot\ \mathsf{ctx}$.

The Goal

We want to prove an internal version of:

Theorem. The homotopy type of a manifold M may be computed as the realization of a certain simplicial set built from the Čech complex of any "good" cover.

The Čech Complex

For $f : X \to Y$, the Čech complex is the simplicial diagram

$$\begin{array}{c} \vdots \\ X \times_Y X \times_Y X \\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow \\ X \times_Y X \\ \downarrow\ \uparrow\ \downarrow \\ X \end{array}$$

Definition. The *Čech complex* $\check{\mathsf{C}}(f)$ of f is its csk_0-image:

$$\check{\mathsf{C}}(f) :\equiv (y : Y) \times \mathsf{csk}_0((x : X) \times (fx = y)).$$

The Čech Complex

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X \times_Y X \times_Y X \\
\downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \\
X \times_Y X \\
\downarrow \; \uparrow \; \downarrow \\
X
\end{array}
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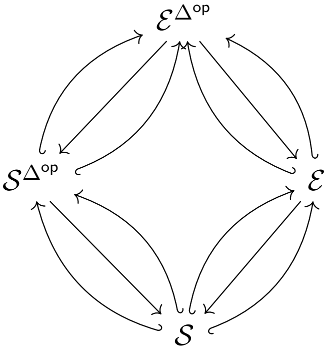
$$\check{\mathsf{C}}(f) :\equiv (y : Y) \times \mathsf{csk}_0((x : X) \times (fx = y)).$$

Proposition. For 0-skeletal X and Y,

$$\check{\mathsf{C}}(f)_n \simeq X \times_Y \cdots \times_Y X \simeq (y : Y) \times ((x : X) \times (fx = y))^{n+1}$$

Proof.

$$\begin{aligned}
&\check{\mathsf{C}}(f)_n \\
&:\equiv \mathsf{sk}_0(\Delta[n] \to \check{\mathsf{C}}(f)) \\
&\equiv \mathsf{sk}_0(\Delta[n] \to (y : Y) \times \mathsf{csk}_0((x : X) \times (fx = y))) \\
&\simeq \mathsf{sk}_0((\sigma : \Delta[n] \to Y) \times ((i : \Delta[n]) \to \mathsf{csk}_0((x : X) \times (fx = \sigma i)))) \\
&\simeq \mathsf{sk}_0((y : Y) \times (\Delta[n] \to \mathsf{csk}_0((x : X) \times (fx = y)))) \\
&\simeq \mathsf{sk}_0((y : Y) \times \mathsf{csk}_0([n] \to (x : X) \times (fx = y))) \\
&\simeq ((u : \mathsf{sk}_0\, Y) \times \mathsf{let}\ y^{\mathsf{sk}_0} := u\ \mathsf{in}\ \mathsf{sk}_0([n] \to (x : X) \times (fx = y))) \\
&\simeq ((y : Y) \times ([n] \to (x : X) \times (fx = y))) \\
&\simeq (y : Y) \times ((x : X) \times (fx = y))^{n+1}
\end{aligned}$$


$\mathcal{E}^{\Delta^{\mathsf{op}}}$
$\mathcal{S}^{\Delta^{\mathsf{op}}}$
$\mathcal{E}$
$\mathcal{S}$

Commuting Cohesions

Add a copy of all the above rules for another annotation ♣.

$$\flat_{\clubsuit}\text{-FORM}\ \frac{\clubsuit \setminus \Gamma \vdash A \ \mathsf{type}}{\Gamma \vdash \flat_{\clubsuit} A \ \mathsf{type}} \qquad \sharp_{\clubsuit}\text{-FORM}\ \frac{\clubsuit\Gamma \vdash A \ \mathsf{type}}{\Gamma \vdash \sharp_{\clubsuit} A \ \mathsf{type}}$$

...

The possible annotations on variables are $\{\varnothing, \heartsuit, \clubsuit, \heartsuit\clubsuit\}$.

$$\text{CTX-EXT}\ \frac{\bullet \subseteq \{\heartsuit, \clubsuit\} \qquad \bullet \setminus \Gamma \vdash A \ \mathsf{type}}{\Gamma, x :_{\bullet} A \ \mathsf{ctx}}$$

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$$\ldots$$

The possible annotations on variables are $\{\varnothing, ♥, ♣, ♥♣\}$.

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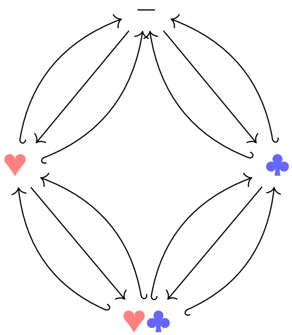
Commuting Cohesions



Proposition. Any lemmas and theorems concerning ($\flat$ and $\sharp$) using no axioms are true also of ($\flat_\heartsuit$ and $\sharp_\heartsuit$) and ($\flat_\clubsuit$ and $\sharp_\clubsuit$).

Lemma. $\flat_\heartsuit$ and $\flat_\clubsuit$ commute.

Proof.

$$\flat_\heartsuit \flat_\clubsuit X \to \flat_\clubsuit \flat_\heartsuit X$$
$$u \mapsto \mathsf{let}\ v^{\flat_\heartsuit} := u\ \mathsf{in}\ (\mathsf{let}\ w^{\flat_\clubsuit} := v\ \mathsf{in}\ w^{\flat_\heartsuit \flat_\clubsuit})$$

and vice versa. □

Lemma. $\sharp_\heartsuit$ and $\sharp_\clubsuit$ commute.

Proof. $v \mapsto v_{\sharp_\heartsuit \sharp_\clubsuit}{}^{\sharp_\heartsuit \sharp_\clubsuit}$ and vice versa. □

Lemma. $\int_\heartsuit$ and $\int_\clubsuit$ commute (when they both exist).

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But not everything commutes with everything!

Definition. If types G and H detect connectivity of ♥ and ♣, we say ♥ and ♣ are *orthogonal* when G is $\flat_{♣}$-modal and H is $\flat_{♥}$-modal.

Lemma. If X is $\int_{♣}$-modal then $\sharp_{♥} X$ is also $\int_{♣}$-modal.

Proof.

$$
\begin{aligned}
&(H \to \sharp_{♥} X) \\
&\simeq \sharp_{♥}(H \to \sharp_{♥} X) \\
&\simeq \sharp_{♥}(\flat_{♥} H \to X) \\
&\simeq \sharp_{♥}(H \to X) && \text{since } H \text{ was assumed } \flat_{♥}\text{-modal} \\
&\simeq \sharp_{♥} X && \text{since } X \text{ is } \textstyle\int_{♣}\text{-modal}
\end{aligned}
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But not everything commutes with everything!

Definition. If types G and H detect connectivity of ♥ and ♣, we say ♥ and ♣ are *orthogonal* when G is $\flat_{\clubsuit}$-modal and H is $\flat_{\heartsuit}$-modal.

Lemma. If X is $\int_{\clubsuit}$-modal then $\sharp_{\heartsuit} X$ is also $\int_{\clubsuit}$-modal.

Proof.

$$
\begin{aligned}
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&\simeq \sharp_{\heartsuit}(H \to \sharp_{\heartsuit} X) \\
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&\simeq \sharp_{\heartsuit} X && \text{since } X \text{ is } \textstyle\int_{\clubsuit}\text{-modal}
\end{aligned}
$$

□

Still if ♥ and ♣ are orthogonal,

Proposition. (♣-crisp $\int_{\heartsuit}$-induction)
$\flat_{\clubsuit}(\flat_{\clubsuit}\int_{\heartsuit} A \to B) \to \flat_{\clubsuit}(\flat_{\clubsuit} A \to B)$ is an equivalence for $\int_{\heartsuit}$-modal B.

Proof.

$$
\begin{aligned}
&\flat_{\clubsuit}(\flat_{\clubsuit}\textstyle\int_{\heartsuit} A \to B) && \\
&\simeq \flat_{\clubsuit}(\textstyle\int_{\heartsuit} A \to \sharp_{\clubsuit} B) && \text{by } \flat_{\clubsuit} \dashv \sharp_{\clubsuit} \\
&\simeq \flat_{\clubsuit}(A \to \sharp_{\clubsuit} B) && \text{by the previous Lemma} \\
&\simeq \flat_{\clubsuit}(\flat_{\clubsuit} A \to B) && \text{by } \flat_{\clubsuit} \dashv \sharp_{\clubsuit}
\end{aligned}
$$

□

Lemma. $\int_{\heartsuit}$ and $\flat_{\clubsuit}$ commute.

Proof. Use the induction principles in both directions. □

Corollary. $\flat_{\heartsuit}$ and $\sharp_{\clubsuit}$ commute.

Assume

- ♥ satisfies the axioms of Real Cohesion, $\int \dashv \flat \dashv \sharp$;
- ♣ satisfies the axioms of Simplicial Cohesion, $\mathsf{re} \dashv \mathsf{sk}_0 \dashv \mathsf{csk}_0$;
- They are orthogonal ($\mathbb{R}$ is 0-skeletal and $\Delta[1]$ is discrete);
- $\int$ is calculated levelwise: $(\eta)_n : X_n \to (\int X)_n$ is itself a $\int$-unit.

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- $\int$ is calculated levelwise: $(\eta)_n : X_n \to (\int X)_n$ is itself a $\int$-unit.

Definition. A *cover* of a 0-skeletal type M is a family $U : I \to (M \to \mathbf{Prop})$ for a discrete 0-skeletal set I so that for every $m : M$ there is merely an $i : I$ with $m \in U_i$.

Definition. A cover is *good* if for any $n : \mathbb{N}$ and any $k : [n] \to I$, the ʃ-shape of

$$\bigcap_{i:[n]} U_{k(i)} :\equiv (m : M) \times ((i : [n]) \to (m \in U_{k(i)})).$$

is a proposition.

Good Covers

Definition. A *cover* of a 0-skeletal type M is a family $U : I \to (M \to \mathbf{Prop})$ for a discrete 0-skeletal set I so that for every $m : M$ there is merely an $i : I$ with $m \in U_i$.

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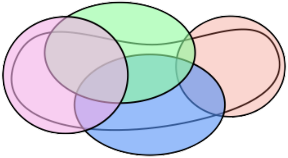

The Projection $\pi : \check{\mathsf{C}}(c) \to \mathsf{csk}_0\, I$

We may assemble a cover into a single surjective map $c : \bigsqcup_{i:I} U_i \to M$, where

$$\bigsqcup_{i:I} U_i :\equiv (i : I) \times (m : M) \times (m \in U_i).$$

Then there is a projection $\pi : \check{\mathsf{C}}(c) \to \mathsf{csk}_0\, I$

$$\begin{array}{c} \vdots \\ C \times_M C \times_M C \\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow \\ C \times_M C \\ \downarrow\ \uparrow\ \downarrow \\ C \end{array} \quad \xmapsto{\ \pi\ } \quad \begin{array}{c} \vdots \\ I \times I \times I \\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow \\ I \times I \\ \downarrow\ \uparrow\ \downarrow \\ I \end{array}$$

Lemma. U is a good cover iff the restriction $\pi : \check{\mathsf{C}}(c) \to \mathsf{im}\,\pi$ is a $\int$-unit.

By an axiom, it suffices to check this on simplices, and we have a convenient description of $\check{\mathsf{C}}(c)_n$

Theorem. $\mathsf{re}\,\mathsf{im}\,\pi \simeq \int M$

Proof. The previous says that $\mathsf{im}\,\pi \simeq \int \check{\mathsf{C}}(c)$, then

$$\mathsf{re}\,\mathsf{im}\,\pi \simeq \mathsf{re} \int \check{\mathsf{C}}(c) \simeq \int \mathsf{re}\,\check{\mathsf{C}}(c) \simeq \int \mathsf{im}\,c \simeq \int M.$$

□

$\mathsf{im}\,\pi$ is a subtype of $\mathsf{csk}_0\,I$. By assumption I is discrete, so $\mathsf{csk}_0\,I$ is discrete, and then $\mathsf{im}\,\pi$ is discrete. So we have exhibited $\int M$ as the realization of a discrete simplicial set.